

Approach of the Generating Functions to the Coherent States for Some Quantum Solvable Models

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May 14, 2014

Abstract

We introduce to this paper new kinds of coherent states for some quantum solvable models: a free particle on a sphere, one-dimensional Calogero-Sutherland model, the motion of spinless electrons subjected to a perpendicular magnetic field B , respectively, in two dimensional flat surface and an infinite flat band. We explain how these states come directly from the generating functions of the certain families of classical orthogonal polynomials without the complexity of the algebraic approaches. We have shown that some examples become consistent with the Klauder-Perelomov and the Barut-Girardello coherent states. It can be extended to the non-classical, q -orthogonal and the exceptional orthogonal polynomials, too. Especially for physical systems that they don't have a specific algebraic structure or involved with the shape invariance symmetries, too.

PACS Nos: 02.30.Gp, 02.20.Sv, 42.50.Dv, 03.65-w, 05.30. - d, 03.65.Fd, 03.65.Ge

1 Reviews and Motivation

According to the pioneering work of Schrödinger [1] coherent states in its general form can be realized as Gaussian wave-function could be constructed from a particular superposition of the wave functions corresponding to the discrete eigenvalues of the harmonic oscillator. They play an important role in quantum optics and provide us with a link between quantum and classical oscillators. Here, it is necessary to emphasize that quantum coherence of states nowadays pervades many branches of physics such as quantum electrodynamics, solid-state physics, and nuclear and atomic physics, from both theoretical and experimental viewpoints. Successfully, these states were applied to some other models considering with their Lie algebra symmetries by Glauber [2], Klauder [3], Sudarshan [4], Barut and Girardello [5] and

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Perelomov [6]. Additionally, for the models with one degree of freedom either discrete or continuous spectra- with no remark on the existence of a Lie algebra symmetry- Gazeau et al proposed new coherent states, which was parametrized by two real parameters [7]. Also, there exist some considerations in connection with coherent states corresponding to the shape invariance symmetries [8].

To construct coherent states, four main different approaches the so-called Schrödinger, Klauder-Perelomov(K-P), Barut-Girardello(B-G), and Gazeau-Klauder(G-K) methods have been found, so the second and the third approaches rely directly on the Lie algebra symmetries and their corresponding generators. Where the first of them are established only by means of generating function- regardless of the Lie algebraic symmetries.

It is worth to mention that the reverse idea is well known. In other words coherent states may usually be led to the generating functions attached to the classical polynomials, too [9]. Using generating function of the Hermite polynomials $H_n(x)$ [10]

$$G(x, t) = e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x), \quad |t| < \infty \quad (1)$$

then with complexification of $t\sqrt{2} \rightarrow z \in \mathbb{C}$, one gets

$$\begin{aligned} G(x, z) &= e^{xz\sqrt{2}-\frac{z^2}{2}} = \pi^{1/4} e^{\frac{x^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \phi_n(x) \\ &= \pi^{1/4} e^{\frac{x^2+|z|^2}{2}} \langle x | z \rangle_{Sch}, \end{aligned} \quad (2)$$

where we have used the notations $\phi_n(x) \left(= \frac{e^{-\frac{x^2}{2}}}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) \right)$ as the complete and orthonormal eigenvectors of the simple harmonic oscillator(SHO), i.e.

$$\int_{-\infty}^{\infty} \phi_n(x) \phi_m(x) dx = \delta_{nm},$$

and $\langle x | z \rangle_{Sch} \left(= e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \phi_n(x) \right)$ as the Schrödinger type of coherent states¹ attached to it, respectively. Clearly, Eq. (2) indicates that using generating function of the classical orthogonal polynomials one can easily found² the coherency of quantum states, i.e.

$$\langle x | z \rangle_{Sch} = \frac{e^{-\frac{x^2+|z|^2}{2}}}{\pi^{1/4}} G(x, z). \quad (3)$$

It, also, serves the following identities

$$\hat{a} \langle x | z \rangle_{Sch} = z \langle x | z \rangle_{Sch}, \quad (4)$$

$$\langle x | z \rangle_{Sch} = e^{z\hat{a}^\dagger - \bar{z}\hat{a}} \phi_0(x), \quad (5)$$

¹ Subscript 'Sch' refers to the Schrödinger and specifies a particular type of quantum states that are called the canonical coherent states, too.

²However, as will be discussed later, this can not be applied to all of the classical orthogonal polynomials.

with respect to the boson creation and annihilation operators \hat{a}^\dagger and \hat{a} . They represent commutators of the Heisenberg Lie algebra, i.e. $[\hat{a}, \hat{a}^\dagger] = 1$, on the Hilbert space of the orthonormal eigenvectors of the harmonic oscillator, $\mathcal{H} := \text{span}\{\phi_n(x) : \int_{-\infty}^{\infty} dx \phi_n(x) \phi_{n'}(x) = \delta_{nn'}\} |_{n=0}^{\infty}$, and realize the usual ladder relations $\hat{a}^\dagger \phi_n = \sqrt{n+1} \phi_{n+1}$ and $\hat{a} \phi_n = \sqrt{n} \phi_{n-1}$. In fact, Eq (3) together with the Eqs. (4) and (5) refer to the fact that the above mentioned approaches the Schrödinger, B-G and K-P come to the same results when applied to the simple harmonic oscillator. It should be noted that these methods have different and distinct results for other physical models, respectively.

Following the extension of the above idea of other classical orthogonal polynomials, we have tried it on several cases including the associated Legendre polynomials $P_l^m(x)$ as well as the associated Laguerre polynomials $L_n^m(x)$ and the associated Bessel functions $B_{n,m}^{q,\beta}$ [45]. Because of the relations between the associated Legendre functions $P_l^m(x)$ and the spherical harmonics $Y_{lm}(\theta, \phi)$, they correspond to the quantum states of many physical problems and have, also, been studied in the framework of the coherent states theory (see, for example, Refs. [12- 24]). Therefore, our motivation in this paper is to make new kind of coherent states as infinite superpositions of the spherical harmonics which are based only on the structure of the corresponding generating function of no remark on the existence of the Lie algebra symmetries. Here, we want to emphasize that the people could not think about constructing of this type of coherent states. The reverse idea is previously discussed and realization of the Klauder-Perelomov types of coherent states as finite superpositions of the spherical harmonics associated to the unitary irreducible representations of the compact algebra $su(2)$ via orbital angular momentum operators is years old, also realization of the Barut-Girardello as well as the Klauder-Perelomov types of coherent states as infinite superpositions of the spherical harmonics corresponding to the infinite-dimensional representations of the Lie algebra $su(1, 1)$ [42], have recently been introduced [22, 23]. Also, Fakhri. et. al established new type of the coherent states in accordance with the new generating functions associated with the spherical harmonics [24]. It should be noted that different generating functions attached to the associated Legendre polynomials of different quantum numbers l and m , for example [11]

$$G_m(x, t) = \frac{(2m)!(1 - x^2)^{m/2}}{2^m m! (1 + t^2 - 2xt)^{m+1/2}} = \sum_{l=0}^{\infty} t^l P_{l+m}^m(x), \quad |t| < 1. \quad (6)$$

Along with the application of the second type of the generating functions (6), we construct new kind of coherent states as infinite superpositions of the spherical harmonics in the section (2). Which is distinct from the the Barut-Girardello as well as the Klauder-Perelomov types of coherent states discussed earlier in Refs [22, 23, 24].

Another part of this document, that will become in section (3), is spent to design and analysis of the coherent states are derived from the generating functions of the associated

Laguerre polynomials L_n^m [10]

$$G_m^+(x, z) = (xz)^{-\frac{m}{2}} J_m(2\sqrt{xz})e^z = \sum_{n=0}^{\infty} \frac{z^n}{(n+m)!} L_n^m(x), \quad m > -1 \quad (7)$$

$$G_m^-(x, z) = \frac{e^{-\frac{xz}{1-z}}}{(1-z)^{m+1}} = \sum_{n=0}^{\infty} z^n L_n^m(x), \quad |z| < 1, \quad (8)$$

$$G_m^0(x, z) = (1+z)^m e^{-xz} = \sum_{n=0}^{\infty} z^n L_n^{m-n}(x), \quad |z| < 1. \quad (9)$$

Because of there exist some quantum physical models whose solvability and square integrability are connected with the associated Laguerre polynomials. The Landau levels, Morse potential, Calogero-Sutherland model, half-oscillator, radial part of 3-D harmonic oscillator and radial part of a hydrogen-like atom are the six examples [25- 30]. They play an important role in the quantum mechanics and have been considered by many authors in the framework of the coherent states theory [25, 31, 32]. We will show three different kinds of the generating functions (7), (8) and (9) lead to the well known Barut-Girardello and the Klauder-Perelomov types of coherent states respectively, already obtained in Refs [31, 32] due to the one dimensional Calogero-Sutherland model as well as two dimensional Landau levels. In other words the generating function technique presented here, come to the same results with the B-G and the K-P methods when applied to the Landau Levels and the Calogero-Sutherland model. It should be noted that in the generating function formalism we don't require an algebraic structure and don't deal with the complexity of algebraic methods.

The last part is devoted to the Bessel functions $B_{l,m}^{(q,\beta)}(x)$. The square-integrable associated Bessel functions can be applied to obtain bound states corresponding to some one-dimensional super-symmetric potential and also some two-dimensional quantum-mechanical models having a Lie algebra symmetry of $su(1, 1)$ [42, 45, 46]. Therefore, exponential generating functions corresponding to the formal power series of associate Bessel functions $B_{l,m}^{(q,\beta)}(x)$ of the same l , the same $l+m$ and the same $l-m$, are important not only from the point of view of mathematical derivation but also from the point of view of physical applications.

2 Coherent states attached to the associated Legendre polynomials $P_l^m(x)$

Let us remind that the spherical harmonics in terms of the polar and azimuthal angles, $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$, can be written as:

$$\langle \theta, \varphi | l, m \rangle := Y_l^m(\theta, \varphi) = (-1)^m \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} e^{im\varphi} P_l^m(\cos \theta), \quad l \in \mathbb{N}_0, \quad -l \leq m \leq l. \quad (10)$$

They constitute an infinite dimensional Hilbert space $\mathcal{H} = L^2(S^2, d\Omega(\theta, \phi))$,

$$\langle l, m | l', m' \rangle := \int_{S^2} Y_l^{m*}(\theta, \varphi) Y_{l'}^{m'}(\theta, \varphi) d\Omega(\theta, \phi) = \delta_{ll'} \delta_{mm'}. \quad (11)$$

The bases of \mathcal{H} are independent spherical harmonics on the sphere S^2 with different values for both indices l and m . The properties of the spherical harmonics are well known and may be found in many texts and papers (see, for example, Refs. [23- 29]). The spherical harmonics with a given l are wave functions of a free particle which has to be placed on the surface of the sphere of a given energy [41]. With the application of angular momentum operators, a given unitary irreducible representation of $so(3) \cong su(2)$ Lie algebra is characterized as the spherical harmonics with a given l . This is the most well-known property of them. It is worth to mention, for decades the spherical harmonics, $Y_{lm}(\theta, \phi)$ have been considered as the representation space of compact Lie algebra $su(2)$ and recently extended by Fakhri to noncompact Lie algebra $su(1, 1)$ [42].

Motivate by this situation, in this paper we will consider the general generating functions $G(\theta, t)$ of the associated Legendre functions P_l^m , are given by [11]

$$\begin{aligned} G(\theta, t) &= \frac{(2m)!}{2^m m!} \frac{(\sin \theta)^m}{(1 + t^2 - 2t \cos \theta)^{m+1/2}} \\ &= \sum_{l=0}^{\infty} t^l P_{m+l}^m(\cos \theta), \quad |t| \leq 1. \end{aligned} \quad (12)$$

The above expression after restoring the Eq. (11), provide us to evaluate furthered sums involving spherical harmonics

$$\frac{(-1)^m e^{im\varphi}}{\sqrt{4\pi}} G(\theta, t) = \sum_{l=0}^{\infty} t^l \sqrt{\frac{(l+2m)!}{l!(2l+2m+1)}} Y_{m+l}^m(\theta, \varphi). \quad (13)$$

Along with substitution of $t \longrightarrow \mathfrak{z} \in \mathbb{C}$, it becomes

$$\frac{(-1)^m e^{im\varphi}}{\sqrt{4\pi}} G(\theta, \mathfrak{z}) = \langle \theta, \varphi | \left(\sum_{l=0}^{\infty} \mathfrak{z}^l \sqrt{\frac{(l+2m)!}{l!(2l+2m+1)}} | m+l, m \rangle \right). \quad (14)$$

We shall show that the expression of right hand sides of the Eq. (16) has all the features of coherent states. For this reason we introduce new kind of coherent states of spherical harmonics as follows

$$|\mathfrak{z}\rangle_m = M_m^{-1/2}(|\mathfrak{z}|) \sum_{l=m}^{\infty} \mathfrak{z}^l \sqrt{\frac{(l+m)!}{(l-m)!(2l+1)}} | l, m \rangle, \quad (15)$$

where $M_m^{-1/2}(|\mathfrak{z}|)$ is chosen so that $|\mathfrak{z}\rangle_m$ is normalized, i.e. ${}_m\langle \mathfrak{z} | \mathfrak{z} \rangle_m = 1$. Due to the orthogonality relation (13) follow that overlapping of two different kinds of these normalized states become nonorthogonal, if $\mathfrak{z}' \neq \mathfrak{z}$, i.e.

$${}_m\langle \mathfrak{z}' | \mathfrak{z} \rangle_m = \frac{(3m)!}{(2m+1)} \frac{(\overline{\mathfrak{z}'\mathfrak{z}})^m}{\sqrt{M_m(|\mathfrak{z}'|)M_m(|\mathfrak{z}|)}} {}_2F_1 \left(\left[3m+1, m+\frac{1}{2} \right], \left[m+\frac{3}{2} \right], \overline{\mathfrak{z}'\mathfrak{z}} \right). \quad (16)$$

Then, $M_m(|\mathfrak{z}|)$ can be calculated to be taken as

$$M_m(|\mathfrak{z}|) = \frac{(3m)!}{(2m+1)} |\mathfrak{z}|^{2m} {}_2F_1 \left(\left[3m+1, m+\frac{1}{2} \right], \left[m+\frac{3}{2} \right], |\mathfrak{z}|^2 \right). \quad (17)$$

From the completeness relation of the Fock sub-space states, resolution of the identity condition

$$\oint_{\mathbb{C}(\mathfrak{z})} |\mathfrak{z}\rangle_m {}_m\langle \mathfrak{z}| d\mu_m(|\mathfrak{z}|) = I_m = \sum_{l=m}^{\infty} |l, m\rangle \langle l, m|, \quad (18)$$

is realized for the states $|\mathfrak{z}\rangle_m$ with $d\mu_m(|\mathfrak{z}|) := \mathfrak{K}_m(|\mathfrak{z}|) \frac{d|\mathfrak{z}|^2}{2} d\phi$, satisfied with the following positive definite and non-oscillating measures on a unit disc:

$$\begin{aligned} \mathfrak{K}_m(|\mathfrak{z}|) &= \frac{M_m(|\mathfrak{z}|)}{\pi(2m-2)!} (1+r^{-2})(1-r^{-2})^{2m-2} \\ &= \frac{(3m)!}{\pi(2m+1)(2m-2)!} \frac{(1+r^2)(1-r^2)^{2m} {}_2F_1\left([3m+1, m+\frac{1}{2}], [m+\frac{3}{2}], r^2\right)}{(1-r^2)^2 r^{2m-2}}, \end{aligned} \quad (19)$$

where we have used the polar coordinate of $\mathfrak{z} = re^{i\phi}$, $0 \leq r \leq 1$, $0 \leq \phi \leq 2\pi$. We want to emphasize, the coherent states $|\mathfrak{z}\rangle_m$ don't obtained through of the B-G nor the K-P methods. We naturally expect that these states will result some new quantum and statistical features.

3 Coherent states attached to the associated Laguerre polynomials $L_m^n(x)$

In this section, we construct some coherent states according to the one dimensional Calogero-Sutherland model and the two dimensional Landau Levels are emerged through of the three different kinds of the generating functions of the associated Laguerre polynomials Eqs. (7), (8) and (9).

• **Coherent states arising from the first generating function $G_m^+(x, z)$ in Eq. (7):** Here, we review the solvability of the Calogero-Sutherland Hamiltonian H^λ on the half-line x

$$H^\lambda = \frac{1}{2} \left[-\frac{d^2}{dx^2} + x^2 + \frac{\lambda(\lambda-1)}{x^2} \right], \quad (20)$$

where, the simple an-harmonic term $\frac{\lambda(\lambda-1)}{x^2}$ refers to the Goldman-Krivchenkov potential [27]. In Refs. [31, 43], it has been shown that the second-order differential operators

$$J_\pm^\lambda := \frac{1}{4} \left[\left(x \mp \frac{d}{dx} \right)^2 - \frac{\lambda(\lambda-1)}{x^2} \right], \quad (21)$$

$$J_3^\lambda := \frac{H^\lambda}{2}, \quad (22)$$

satisfy the standard commutation relations of $su(1, 1)$ Lie algebra as follows

$$[J_+^\lambda, J_-^\lambda] = -2J_3^\lambda, \quad [J_3^\lambda, J_\pm^\lambda] = \pm J_\pm^\lambda. \quad (23)$$

Also, product the unitary and positive-integer irreps of $su(1, 1)$ Lie algebra as

$$J_+^\lambda |n-1, \lambda\rangle = \sqrt{n \left(n + \lambda - \frac{1}{2} \right)} |n, \lambda\rangle, \quad (24)$$

$$J_-^\lambda |n, \lambda\rangle = \sqrt{n \left(n + \lambda - \frac{1}{2} \right)} |n-1, \lambda\rangle, \quad (25)$$

$$J_3^\lambda |n, \lambda\rangle = \left(n + \frac{\lambda}{2} + \frac{1}{4} \right) |n, \lambda\rangle. \quad (26)$$

We assume that the set of states described above, form complete and orthonormal basis of an infinite dimensional Hilbert space, i.e.

$$\begin{aligned} \mathcal{H}^\lambda &:= \text{span}\{|n, \lambda\rangle | \langle n, \lambda | m, \lambda \rangle = \delta_{nm}\}_{n=0}^\infty, \\ \langle x | n, \lambda \rangle &:= (-1)^n \sqrt{\frac{2\Gamma(n+1)}{\Gamma(n+\lambda+\frac{1}{2})}} x^\lambda e^{-\frac{x^2}{2}} L_n^{\lambda-\frac{1}{2}}(x^2), \quad \lambda > \frac{-1}{2}, \end{aligned} \quad (27)$$

where $L_n^{\lambda-\frac{1}{2}}(x)$ denotes the associated Laguerre polynomials [10]. Along with the orthogonality of the associated Laguerre polynomials, the orthogonality relation of the basis of \mathcal{H}^λ reads

$$\langle n, \lambda | m, \lambda \rangle := \frac{2n!}{\Gamma(n+\lambda+\frac{1}{2})} \int_0^\infty x^{2\lambda} e^{-x^2} L_n^{\lambda-\frac{1}{2}}(x^2) L_m^{\lambda-\frac{1}{2}}(x^2) dx = \delta_{nm}. \quad (28)$$

Using Eqs. (7) and (27), one gets

$$\begin{aligned} \sqrt{2} x^\lambda e^{-\frac{x^2}{2}} G_{\lambda-\frac{1}{2}}^+(x^2, z) &= \sqrt{2} x^{\frac{1}{4}} z^{\frac{1-2\lambda}{4}} e^{z-\frac{x^2}{2}} J_{\lambda-\frac{1}{2}}(2x\sqrt{z}) \\ &= \sum_{n=0}^\infty \frac{(-z)^n}{\sqrt{n! \Gamma(n+\lambda+\frac{1}{2})}} \langle x | n, \lambda \rangle, \quad \lambda > -\frac{1}{2}. \end{aligned} \quad (29)$$

Obviously, R.H.S of (29) is proportional to the Barut-Giradello coherent states for the Calogero-Sutherland model already discussed in [31] (see Eq. (6) therein) i.e.

$$\sum_{n=0}^\infty \frac{(-z)^n}{\sqrt{n! \Gamma(n+\lambda+\frac{1}{2})}} \langle x | n, \lambda \rangle \equiv \langle x | -z \rangle_{BG}^\lambda, \quad (30)$$

and satisfies an eigenvalue equation with respect to the lowering operator J_-^λ . We conclude that the generating function of the associated Laguerre polynomials (7) is proportional to the Barut-Giradello coherent states for the Calogero-Sutherland model up to a normalization coefficient, i.e.

$$\langle x | -z \rangle_{BG}^\lambda \equiv \sqrt{2} x^\lambda e^{-\frac{x^2}{2}} G_{\lambda-\frac{1}{2}}^+(x^2, z). \quad (31)$$

It indicates that the generating function $G_m^+(x, z)$ in Eq. (7), come to the B-G coherent states when applied to the Calogero-Sutherland model.

• **Coherent states arising from the second generating function $G_m^-(x, z)$, Eq. (8):** Similarity to what we have performed above, one can show that

$$\begin{aligned}\sqrt{2}x^\lambda e^{-\frac{x^2}{2}} G_{\lambda-\frac{1}{2}}^-(x^2, z) &= \sqrt{2} \frac{x^\lambda e^{-\frac{x^2}{2}(1+2\frac{z}{1-z})}}{(1-z)^{\lambda+\frac{1}{2}}} \\ &= \sum_{n=0}^{\infty} (-z)^n \sqrt{\frac{\Gamma(n+\lambda+\frac{1}{2})}{n!}} \langle x|n, \lambda \rangle\end{aligned}\quad (32)$$

$$\equiv \langle x | -z \rangle_{KP}^\lambda. \quad (33)$$

Clearly, it illustrates a correspondence between the generating function of the associated Laguerre polynomials (8) and the Klauder-Perelomve coherent states for the Calogero-Sutherland model(Eq. (11) in Ref. [29]).

• **Coherent states arising from the third generating function $G_m^0(x, z)$ in Eq. (9):** Likewise, it is easy to see that the third generating function $G_m^0(x, z)$

$$G_m^0(x, z) = (1+z)^m e^{-xz} = \sum_{n=0}^{\infty} z^n L_n^{m-n}(x) = z^m \sum_{n=-m}^{\infty} z^n L_{n+m}^{-n}(x), \quad |z| < 1, \quad (34)$$

with respect to the polar coordinate, $0 < r < \infty, 0 \leq \varphi < 2\pi$, representation of Landau levels in terms of the associated Laguerre functions

$$\langle r, \varphi | n+m, -n \rangle = \sqrt{\frac{(n+m)! \left(\frac{M\omega}{2\hbar}\right)^{-n+1}}{\pi n!}} r^{-n} e^{-in\varphi} e^{-\frac{M\omega}{4\hbar} r^2} L_{n+m}^{(-n)}\left(\frac{M\omega r^2}{2\hbar}\right), \quad (35)$$

becomes

$$\begin{aligned}\sqrt{\frac{1}{2\pi}} z^{-m} e^{-M\omega r^2/4\hbar} G_m(M\omega r^2/2\hbar, z) &= \left(1 + \frac{1}{z}\right)^m e^{-\frac{M\omega r^2}{4\hbar}(1+2z)} \\ &= \sum_{n=-m}^{\infty} \left(z e^{i\phi} \sqrt{\frac{M\omega r^2}{2\hbar}}\right)^n \sqrt{\frac{m!}{(m+n)!}} \langle r, \varphi | n+m, -n \rangle.\end{aligned}\quad (36)$$

Here, the Landau levels are related to the symmetric-gauge Landau Hamiltonian³ corresponding to the motion of an electron on the flat surface in the presence of an unified magnetic field, B_{ext} , in the positive direction of z axis [25, 32], i.e.

$$H|n, m\rangle = \hbar\omega \left(n + \frac{1}{2}\right) |n, m\rangle, \quad (37)$$

$$H = \hbar\omega \left(a^\dagger a + \frac{1}{2}\right) = \hbar\omega \left(b^\dagger b + \frac{1}{2}\right) - \omega L_3, \quad (38)$$

³The Landau Hamiltonian, H , has an infinite-fold degeneracy on the Landau levels, that is in which Landau cyclotron frequency is expressed in terms of the value of the electron charge, its mass, the magnetic field strength B_{ext} and also the velocity of light as $\omega = \frac{eB_{ext}}{Mc}$. Here m is an integer and n a nonnegative integer, subject to the constraint $m \geq -n$.

where $L_3 = -i\frac{\partial}{\partial\varphi}$ and

$$a = -e^{i\varphi}\sqrt{\frac{\hbar}{2M\omega}}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\varphi} + \frac{M\omega}{2\hbar}r\right), \quad a^\dagger = e^{-i\varphi}\sqrt{\frac{\hbar}{2M\omega}}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\varphi} - \frac{M\omega}{2\hbar}r\right), \quad (39)$$

$$b = e^{-i\varphi}\sqrt{\frac{\hbar}{2M\omega}}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\varphi} + \frac{M\omega}{2\hbar}r\right), \quad b^\dagger = -e^{i\varphi}\sqrt{\frac{\hbar}{2M\omega}}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\varphi} - \frac{M\omega}{2\hbar}r\right). \quad (40)$$

They form two separate copies of Weyl-Heisenberg algebra,

$$[a, a^\dagger] = 1, [b, b^\dagger] = 1, [a, b^\dagger] = [a^\dagger, b] = [a, b] = [a^\dagger, b^\dagger] = 0, \quad (41)$$

with the unitary representations as

$$a|n, m\rangle = \sqrt{n}|n-1, m+1\rangle, \quad a^\dagger|n-1, m+1\rangle = \sqrt{n}|n, m\rangle, \quad (42)$$

$$b|n, m\rangle = \sqrt{n+m}|n, m-1\rangle, \quad b^\dagger|n, m-1\rangle = \sqrt{n+m}|n, m\rangle. \quad (43)$$

Also, they are complex conjugate of each other with respect to the following orthogonality integration over the entire plane,

$$\langle n, m|n', m'\rangle := \int_{\varphi=0}^{2\pi} \int_{r=0}^{\infty} \langle r, \varphi|n, m\rangle^* \langle r, \varphi|n', m'\rangle r dr d\varphi = \delta_{nn'} \delta_{mm'}. \quad (44)$$

Applying the annihilation operator a on the *R.H.S* of the Eq. (36), then using the ladder relation (42) one can find that it satisfies an *eigenvalue equation*, which implies a proportionality among these coherent states emerged through of the generating function, $G_m^0(x, z)$, of the associated Laguerre polynomials and the normalized Weyl-Heisenberg coherent states, $|ze^{i\phi}\sqrt{\frac{M\omega r^2}{2\hbar}}\rangle_n$, i.e.

$$\sqrt{\frac{1}{2\pi}} z^{-m} e^{-M\omega r^2/4\hbar} G_m(M\omega r^2/2\hbar, z) \equiv \langle r, \varphi | ze^{i\phi}\sqrt{\frac{M\omega r^2}{2\hbar}}\rangle_n^b. \quad (45)$$

Relevant characteristics such as resolution of the identity condition as well as their statistical properties were investigated in Refs. [25, 32].

4 Coherent states attached to the associated Bessel functions $B_{n,m}^{\mu,\nu}(x)$

The Model:

Consider a spinless electron with charge $e < 0$ and an effective mass μ moving on an infinite flat band of the presence of a Morse-like perpendicular magnetic fields directed in the negative z -direction $B = -B_0 e^{-\frac{2\pi}{a_0}x} \hat{k}$ with $B_0 > 0$ as a constant magnetic strength and a_0 as the width of the band. It leads to the following Morse-like vector potential [46]

$$A_x = \frac{i\pi\hbar c}{ea_0} - i\frac{a_0 B_0}{2\pi} e^{-\frac{2\pi}{a_0}x}, \quad A_y = \frac{a_0 B_0}{2\pi} e^{-\frac{2\pi}{a_0}x}, \quad A_z = 0, \quad (46)$$

in which c is the velocity of electromagnetic waves in the vacuum. For an electron of effective mass μ , moving on the infinite flat band in the presence of the vector potential (46), the time-independent Schrödinger wave equation

$$\frac{1}{2\mu} \left[\left(p_x - \frac{e}{c} A_x \right)^2 + \left(p_y - \frac{e}{c} A_y \right)^2 \right] \psi = E\psi,$$

can be written in terms of the variables $\xi = e^{\frac{2\pi}{a_0}x}$ ($0 < \xi < \infty$) and $-\frac{a_0}{2} < y < \frac{a_0}{2}$ as

$$\left[-\xi^2 \frac{\partial^2}{\partial \xi^2} + \left(\frac{eB_0 a_0^2}{2\pi^2 \hbar c} - 2\xi \right) \frac{\partial}{\partial \xi} - \frac{a_0^2}{4\pi^2} \frac{\partial^2}{\partial y^2} + i \frac{eB_0 a_0^3}{4\pi^3 \hbar c} \frac{\partial}{\partial y} - \frac{1}{4} \right] \psi = \frac{2\mu a_0^2}{4\pi^2 \hbar^2} E \psi. \quad (47)$$

The periodic boundary condition in the y -direction requires that the wave function ψ is separated into $\psi = e^{\frac{2i\pi}{a_0}my} \psi(\xi)$. Hence, $\psi(\xi)$ satisfies the following differential equation:

$$\xi^2 \frac{d^2 \psi(\xi)}{d\xi^2} + \left(2\xi - \frac{eB_0 a_0^2}{2\pi^2 \hbar c} \right) \frac{d\psi(\xi)}{d\xi} - \left(m^2 - \frac{1}{4} + \frac{eB_0 a_0^2}{2\pi^2 \hbar c} \frac{m}{\xi} - \frac{2\mu a_0^2}{4\pi^2 \hbar^2} E \right) \psi(\xi) = 0. \quad (48)$$

Which can be compared by the associated Bessel differential equation, and results

$$\psi(\xi) = B_{l,m}^{0,\beta}(\xi). \quad (49)$$

Here, $B_{l,n}^{(q,\beta)}(x)$ refer to the Bessel functions, with $\beta = -\frac{eB_0 a_0^2}{2\pi^2 \hbar c}$. Therefore, the square integrable solutions can be obtained as follow

$$|l, m\rangle := \psi_{l,m}(x, y) = \frac{\beta}{\sqrt{2\pi}} e^{\frac{2i\pi}{a_0}my} B_{l,m}^{0,\beta}(e^{\frac{2\pi}{a_0}x}), \quad (50)$$

They form an orthonormal set with respect to the integer index m ,

$$\langle l, m | l', m' \rangle := \int_{-\frac{a_0}{2}}^{\frac{a_0}{2}} \int_{-\infty}^{\infty} \psi_{l,m}^*(x, y) \psi_{l',m'}(x, y) e^{\frac{2\pi}{a_0}x - \beta e^{-\frac{2\pi}{a_0}x}} dx dy = \delta_{ll'} \delta_{mm'}. \quad (51)$$

We also find that the allowed energies of the electron are quantized as a positive quadratic function of both quantum numbers l and m :

$$E_{l,m} = \frac{2\pi^2 \hbar^2}{\mu a_0^2} \left(m - l - \frac{1}{2} \right) \left(m + l + \frac{1}{2} \right). \quad (52)$$

Therefore, the energy values increase in decreasing width a_0 , without increasing the strength of magnetic fields. In addition, according to our considerations, there is no degeneracy in the case where $(2m - 2l - 1)(2m + 2l + 1)$ is a prime number and two folds in other cases. Also, the linear spectrum of the Landau levels is obtained as a limiting case of $a_0 \rightarrow \infty$.

It has been shown that the square integrable pure states realize representations of $su(1, 1)$ algebra via the quantum number n corresponding to the linear momentum in the y -direction. All of the lowest states of the $su(1, 1)$ representations minimize uncertainty relation and the minimizing of their second and third states is transformed to that of the Landau levels in the limit $a_0 \rightarrow \infty$. The compact forms of the Barut-Girardello coherent states corresponding

to Irreducible representation of $su(1, 1)$ algebra and their positive definite measures on the complex plane are also calculated [46].

What we do, here, is to consider the generating functions for given $l + m$ and $q = 0$, the case the sequences are increasing from l . the sequences are increasing with respect to l . Due to whether $n := -l - m - 1$ is odd or even, i.e. $n = 2k + 1$ or $n = 2k$, the highest functions are $B_{-k-1, \frac{q}{2}-k-1}^{(q, \beta)}(x)$ and $B_{-k, -k-1}^{(0, \beta)}(x)$, respectively. These functions lie on the lines $m = l$ and $m = l - 1$ of figure 1 in [45], respectively. Therefore, it is obvious that the terminology of highest functions has been devoted to the associated Bessel functions $B_{l, m}^{(0, \beta)}(x)$ with the most value of m .

• We suppose that n is even, i.e. $n = 2k$. For a given value of k , the generating functions corresponding to the second-type series will result an appropriate infinite sequence of the associated Bessel functions

$$\begin{aligned} G_{n=2k}(x, t) &= \frac{x^{-k}}{2\sqrt{xt}} \left[\left(1 - \sqrt{xt}\right)^{2k} e^{\beta\sqrt{\frac{t}{x}}} - \left(1 + \sqrt{xt}\right)^{2k} e^{-\beta\sqrt{\frac{t}{x}}} \right] \\ &= \sum_{m=0}^{\infty} \frac{t^m}{(2m+1)!} \frac{B_{m-k, -m-k-1}^{(0, \beta)}(x)}{a_{m-k, -m-k-1}(0, \beta)}, \end{aligned} \quad (53)$$

with

$$a_{l, m}(0, \beta) = \begin{cases} \frac{(-1)^{-m}\beta^{-l}}{\sqrt{(l-m)!(-l-m-1)!}} & \text{if } m \leq l < 0, \\ \frac{(-1)^{-l-m-1}\beta^{-l-1}}{\sqrt{(l-m)!(-l-m-1)!}} & \text{if } 0 \leq l \leq -m-1. \end{cases}$$

It should be noticed that $G_{n=2k}(x)$ is summed over the parameter m for $l + m = -2k - 1$. Thereafter, using x -coordinate representation of the Bessel functions as well as the relation (50), it yields

$$\frac{\beta e^{\frac{-2i\pi}{a_0}(k+1)y}}{\sqrt{2\pi(2k)!}} G_{2k}(x, t) = \langle x | \left(\sum_{m=0}^{\infty} \frac{\left(\beta t e^{\frac{2i\pi}{a_0}y}\right)^m}{\sqrt{(2m+1)!}} | m - k, -m - k - 1 \rangle \right). \quad (54)$$

Where the Fock states $| m - k, -m - k - 1 \rangle$ will require the following completeness and square integrability conditions, respectively, as follows:

$$\sum_{m=0}^{\infty} | m - k, -m - k - 1 \rangle \langle m' - k, -m' - k - 1 | = I_{2k}, \quad (55)$$

$$\begin{aligned} \langle m - k, -m - k - 1 | m' - k, -m' - k - 1 \rangle &:= \\ &\int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \psi_{m-k, -m-k-1}^*(x, y) \psi_{m'-k, -m'-k-1}(x, y) e^{-\beta e^{-\frac{2\pi}{a_0}x}} dx dy = \delta_{mm'}. \end{aligned} \quad (56)$$

Here, I_{2k} refers to the identity(projection) operators on the separable infinite dimensional Hilbert sub-spaces $\mathcal{H}_{2k} := \text{span}\{| m - k, -m - k - 1 \rangle\}$. One can show that the orthogonality relation (56) can be readily obtained with the aid of the generating function (54). This provides us the corresponding CSs to be of the form

$$| \mathfrak{z} \rangle_{2k} \left(= | \beta t e^{\frac{2i\pi}{a_0}y} \rangle \right) := \sqrt{\frac{|\mathfrak{z}|}{\sinh(|\mathfrak{z}|)}} \sum_{m=0}^{\infty} \frac{\mathfrak{z}^m}{\sqrt{(2m+1)!}} | m - k, -m - k - 1 \rangle \quad (57)$$

,and can be regarded as an infinite series of solutions to the Hamiltonian of a spinless electron moving on an infinite flat band of the presence of a Morse-like perpendicular magnetic fields. Here, $\mathfrak{z} = \beta t e^{\frac{2i\pi}{a_0} y}$ as an arbitrary complex variable and coefficients $\sqrt{\frac{|\mathfrak{z}|}{\sinh(|\mathfrak{z}|)}}$ are included to achieve the normalization condition. It is straightforward that, such a superposition contains all the required features of a coherent state. In other word, one can show that these normalized states form an overcomplete system in the separable Hilbert sub-spaces \mathcal{H}_{2k} . Indeed, they solve the identity,

$$\oint_{\mathbb{C}(\mathfrak{z})} |\mathfrak{z}\rangle_{2k} {}_{2k}\langle \mathfrak{z}| d\mu_k(|\mathfrak{z}|) = I_{2k}$$

in terms of an acceptable measure $d\mu(|\mathfrak{z}|) := \mathfrak{K}(|\mathfrak{z}|) \frac{d|\mathfrak{z}|^2}{2} d\phi$ on the whole complex plane, with

$$\mathfrak{K}(|\mathfrak{z}|) = \frac{e^{-\mathfrak{z}}}{2\pi|\mathfrak{z}|} \sinh(|\mathfrak{z}|). \quad (58)$$

• Similar results can be found on the generating function $G_{n=2k+1}(x, t)$ which is summed over the parameter m for $l + m = -2k - 2$ and relates to the second-type sequences as an infinite sequences of the associated Bessel functions

$$\begin{aligned} G_{n=2k+1}(x, t) &= \frac{x^{-k-1}}{2} \left[\left(1 - \sqrt{xt}\right)^{2k+1} e^{\beta\sqrt{\frac{t}{x}}} + \left(1 + \sqrt{xt}\right)^{2k+1} e^{-\beta\sqrt{\frac{t}{x}}} \right] \\ &= \sum_{m=0}^{\infty} \frac{t^m}{(2m)!} \frac{B_{m-k-1, -m-k-1}^{(0, \beta)}(x)}{a_{m-k-1, -m-k-1}(0, \beta)}. \end{aligned} \quad (59)$$

One can show that this leads to the following normalized coherent vector

$$|\mathfrak{z}\rangle_{2k+1} = \frac{1}{\sqrt{\cosh(|\mathfrak{z}|)}} \sum_{m=0}^{\infty} \frac{\mathfrak{z}^m}{\sqrt{(2m)!}} |m - k - 1, -m - k - 1\rangle, \quad (60)$$

where we have used the orthogonality relation

$$\begin{aligned} \langle m - k - 1, -m - k - 1 | m' - k - 1, -m' - k - 1 \rangle := \\ \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \psi_{m-k-1, -m-k-1}^*(x, y) \psi_{m'-k-1, -m'-k-1}(x, y) e^{-\beta e^{-\frac{2\pi}{a_0} x}} dx dy = \delta_{mm'}. \end{aligned} \quad (61)$$

Using the completeness relation $\sum_{m=0}^{\infty} |m - k - 1, -m - k - 1\rangle \langle m' - k - 1, -m' - k - 1| = I_{2k+1}$, it is found that the relation

$$\oint_{\mathbb{C}(\mathfrak{z})} |\mathfrak{z}\rangle_{2k+1} {}_{2k+1}\langle \mathfrak{z}| d\mu_k(|\mathfrak{z}|) = I_{2k+1}$$

is satisfied, through the following positive definite measure

$$d\mu_k(|\mathfrak{z}|) = \frac{e^{-\mathfrak{z}}}{2\pi} \cosh(|\mathfrak{z}|) \frac{d|\mathfrak{z}|^2}{2} d\phi \quad |\mathfrak{z}| \in [0, \infty), \quad 0 \leq \phi \leq 2\pi. \quad (62)$$

5 Discussion and Outlooks

We construct new and generalized coherent states associated to the one and two dimensional quantum solvable models and study some of their mathematical properties. We discuss an algorithm that leads to superposition of quantum states play an important role in quantum physics as coherent states. Despite of well know techniques that are based on the group and representation theory, this is only introduces approaches will focus on generating functions. It results new and different types of coherent states corresponding to the Legendre polynomials and the Bessel functions too, which is discussed for first time then expects to explore some unrevealed features. Our formalism allows us to calculate new kind of coherent states especially for physical systems that they don't have a specific algebraic structure or involved with the shape invariance symmetries, too. Finally, it can be applied to some non-classical and q-deformed polynomials of other quantum systems and will be reported in a future work.

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